

# PERTURBATION OF BAUM-BOTT RESIDUES

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**ABSTRACT.** We prove that Baum-Bott residues vary continuously under smooth deformations of holomorphic foliations. This provides an effective way to compute residues.

## 1. INTRODUCTION

A holomorphic foliation  $\mathcal{F}$  on a complex manifold  $M$  is known to produce a “holomorphic action”, as discovered by P. Baum and R. Bott in [3], on the virtual bundle  $TM/\mathcal{F}$ . Such a partial holomorphic action provides a holomorphic connection for the bundle  $TM/\mathcal{F}$  along  $\mathcal{F}$  outside the singularities of  $\mathcal{F}$  and thus produces localization of sufficiently high degree classes of  $TM/\mathcal{F}$  around the singularities of  $\mathcal{F}$ . Such localizations are called “Baum-Bott residues” (see [3, Thm. 2], [10, Ch.VI, Thm. 3.7]). When the singularity is isolated the Baum-Bott residue can be expressed in terms of a Grothendieck residue (see [3, (0.6)]). When the singular set is non-isolated in some cases some formulas are available (see [3, Thm. 3] and [4]) but, in general, explicit computation of the residues is rather difficult.

The aim of the present paper is to study the behavior of the Baum-Bott residues under smooth deformations. This provides an effective tool for computing residues explicitly.

More in details, we consider a smooth deformation of a complex manifold. This is essentially a smooth fibration over a smooth manifold, whose fibers are complex manifolds (see Section 2). On each such a fiber we consider a holomorphic foliation which varies smoothly (see Section 3). We prove that the Baum-Bott residues (when taken together suitably) vary continuously under smooth deformations.

We state here a simple consequence of our main Theorem 5.5 for the case of classes of top degree, referring the reader to Section 5 for the general case. Thus, let  $P$  be a real manifold, the “parameter space”. Let  $\widetilde{M} := \{M_t\}_{t \in P}$ , be a deformation of complex manifolds of dimension  $n$ . Let  $\widetilde{\mathcal{F}} := \{\mathcal{F}_t\}$  be a deformation of holomorphic foliations on  $M_t$ . Then  $\widetilde{\mathcal{F}}$  defines naturally a smooth foliation on  $\widetilde{M}$  (see Section 3).

Suppose the singular set  $S_{t_0}$  of  $\mathcal{F}_{t_0}$  in  $M_{t_0}$  is compact and connected. The analytic set  $S_{t_0}$  is contained in a connected component in  $\widetilde{M}$  of the singular set of the smooth foliation  $\widetilde{\mathcal{F}}$ , and we denote by  $S_t$  the intersection of such component with  $M_t$ . The set  $S_t$  is contained in the

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singular set of  $\mathcal{F}_t$  but in general may not be connected. Thus, we let  $S_t = \cup S_t^\lambda$  be the connected components decomposition of  $S_t$ .

**Theorem 1.1.** *Suppose that  $S_t$  is compact for all  $t \in P$ . Let  $\varphi$  be a homogeneous symmetric polynomial of degree  $n$  and denote by  $BB_\varphi(\mathcal{F}_t; S_t^\lambda)$  the Baum-Bott residue of  $\mathcal{F}_t$  at  $S_t^\lambda$ . Then*

$$\lim_{t \rightarrow t_0} \sum_{\lambda} BB_\varphi(\mathcal{F}_t; S_t^\lambda) = BB_\varphi(\mathcal{F}_{t_0}; S_{t_0}).$$

A general version of the previous theorem is Theorem 5.5, whose proof is contained in Sections 4 and 5. The rough idea of the proof is to define a special connection on the regular part of the bundle  $T\widetilde{M}/\widetilde{\mathcal{F}}$  such that on each fiber  $M_t$  defines the special connection given by the Baum-Bott action and see the residues as the integral of a smooth form on  $\widetilde{M}$  along the fibers.

In Section 6 we give an explicit example of the previous result. In particular, aside from explicit computation, the example shows that if the residues in the same connected component of  $\widetilde{M}$  are not taken together, continuity is lost.

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## 2. DEFORMATION OF MANIFOLDS

The theory of deformation of complex structures was first systematically developed by K. Kodaira and D. C. Spencer [6], here we recall the basic material relevant for our needs.

**Definition 2.1.** A *deformation of manifolds* is a triple  $(\widetilde{M}, P, \pi)$ , where  $P$  is a  $C^\infty$  manifold of real dimension  $s$ , called the *parameter space*,  $\widetilde{M}$  is a  $C^\infty$  manifold of real dimension  $2n + s$ , called the *ambient manifold*, and  $\pi : \widetilde{M} \rightarrow P$  is a surjective  $C^\infty$  map such that there exists a covering  $\{U_\alpha\}$  (called an *adapted deformation coordinates covering* of  $\widetilde{M}$ ) with the following properties:

- (1) for each  $\alpha$ , the open set  $U_\alpha$  is diffeomorphic to  $D \times V$ , where  $D$  is an open set of  $\mathbb{C}^n$  and  $V$  is an open set of  $\mathbb{R}^s$ , with coordinates  $(z_1^\alpha, \dots, z_n^\alpha, t_1^\alpha, \dots, t_s^\alpha)$ ,
- (2)  $\pi(U_\alpha)$  is diffeomorphic to  $V$  and  $\pi$  is compatible with the projection  $D \times V \rightarrow V$ ,
- (3) on  $U_\alpha \cap U_\beta \neq \emptyset$  we may express as

$$(2.1) \quad \begin{cases} z_i^\beta = z_j^\beta(z^\alpha, t^\alpha) & i = 1, \dots, n \\ t_j^\beta = t_j^\beta(t^\alpha) & j = 1, \dots, s \end{cases}$$

and, for each fixed  $t^\alpha$ , the map  $z^\alpha \mapsto z^\beta(z^\alpha, t^\alpha)$  is holomorphic.

For  $t \in P$  we let  $M_t := \pi^{-1}(t)$  be the fiber over  $t$ . By definition the fibers  $M_t$ , for  $t \in P$ , are complex manifolds. In particular we can define the sheaf  $\widetilde{\mathcal{O}}_{\widetilde{M}}$  of  $C^\infty$  functions on  $\widetilde{M}$  such that  $f \in \widetilde{\mathcal{O}}_{\widetilde{M}}(U)$  if for all  $x \in U$ ,  $f|_{\pi^{-1}(\pi(x))} \in \mathcal{O}_{\pi^{-1}(\pi(x))}(U \cap \pi^{-1}(\pi(x)))$ .

**Remark 2.2.** Let  $U_\alpha \subset \widetilde{M}$  be a coordinate chart of an adapted coordinate covering for  $\widetilde{M}$ . A function  $f$  belongs to  $\widetilde{\mathcal{O}}_{\widetilde{M}}(U_\alpha)$  if and only if  $f(z_\alpha, t_\alpha)$  is a  $C^\infty$  function such that  $f(\cdot, t_\alpha)$  is holomorphic (note that this is well defined by (2.1)).

**Definition 2.3.** Let  $E$  be a  $C^\infty$  complex vector bundle of rank  $r$  over  $\widetilde{M}$ . We say that  $E$  is an  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -(vector) bundle if there exists a trivializing atlas  $\{U_\alpha\}$  for  $E$ , with frames  $\{e_1^\alpha, \dots, e_r^\alpha\}$  for  $E|_{U_\alpha}$ , such that the transition matrices with respect to those frames have entries which are local sections of  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ . Such frames  $\{e_1^\alpha, \dots, e_r^\alpha\}$  are called  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -frames.

Given an  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -bundle  $E$ , we denote by  $\widetilde{\mathcal{O}}_{\widetilde{M}}(E)$  the  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -module of  $\widetilde{\mathcal{O}}_{\widetilde{M}}$  sections of  $E$ . Namely,  $s \in \widetilde{\mathcal{O}}_{\widetilde{M}}(E)(U)$  is a  $C^\infty$  section of  $E$  over the open set  $U \subset \widetilde{M}$  such that in any  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -frame  $\{e_1^\alpha, \dots, e_r^\alpha\}$  over  $U_\alpha$  with  $U_\alpha \cap U \neq \emptyset$  the section  $s$  is given by

$$s(z^\alpha, t^\alpha) = \sum_{j=1}^r f_j^\alpha(z^\alpha, t^\alpha) e_j^\alpha, \quad f_j^\alpha \in \widetilde{\mathcal{O}}_{\widetilde{M}}(U_\alpha \cap U).$$

Let  $T_{\mathbb{R}}\pi := \ker \pi_*$ . Since the fibers of the fibration  $\pi : \widetilde{M} \rightarrow P$  are holomorphic, we can define the complex vector bundles

$$T\pi := \bigcup_{x \in \widetilde{M}} T_x \pi^{-1}(\pi(x)), \quad \overline{T\pi} := \bigcup_{x \in \widetilde{M}} \overline{T_x \pi^{-1}(\pi(x))}.$$

Local frames for  $T\pi$  and  $\overline{T\pi}$  in an adapted deformation coordinates covering are given respectively by  $\{\frac{\partial}{\partial z_j^\alpha}\}_{j=1, \dots, n}$  and  $\{\frac{\partial}{\partial \bar{z}_j^\alpha}\}_{j=1, \dots, n}$  and

$$T_{\mathbb{R}}\pi \otimes \mathbb{C} = T\pi \oplus \overline{T\pi}.$$

Using an adapted deformation coordinates covering, by (2.1), it is easy to see that  $T\pi$  is an  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -vector bundle over  $\widetilde{M}$ . Moreover, it has a natural structure of  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -Lie algebra, namely, using local coordinates, one can easily see that if  $v, w \in \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)(U)$  then

$$[v, w] \in \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)(U).$$

### 3. DEFORMATION OF FOLIATIONS

Deformations of holomorphic foliations, especially from the point of view of moduli spaces, have been studied by a number of authors (e.g. [5], [8], [9]). Here we consider  $C^\infty$  families of singular holomorphic foliations.

Let  $\mathcal{S}$  be an  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -module. We say that  $\mathcal{S}$  is *coherent* if, for each point  $x \in \widetilde{M}$  there exists an open neighborhood  $U \subset \widetilde{M}$  and two integers  $p, q \geq 0$  such that

$$(3.1) \quad \widetilde{\mathcal{O}}_{\widetilde{M}}|_U^p \xrightarrow{\varphi} \widetilde{\mathcal{O}}_{\widetilde{M}}|_U^q \longrightarrow \mathcal{S}|_U \rightarrow 0,$$

is an exact sequence of  $\widetilde{\mathcal{O}}_{\widetilde{M}}|_U$ -modules.

**Definition 3.1.** Let  $(\widetilde{M}, P, \pi)$  be a deformation of manifolds. A coherent  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -submodule  $\widetilde{\mathcal{F}}$  of  $\widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)$  such that  $[\widetilde{\mathcal{F}}, \widetilde{\mathcal{F}}] \subset \widetilde{\mathcal{F}}$  is called a *deformation of foliations*.

Given a deformation of foliations  $\widetilde{\mathcal{F}}$  on a deformation of manifolds  $(\widetilde{M}, P, \pi)$ , we denote by  $\mathcal{C}_P^\infty$  the sheaf of germs of complex valued smooth functions on  $P$ , and for each  $t \in P$ , by  $\mathcal{I}_t := \{f \in \mathcal{C}_P^\infty : f(t) = 0\}$  the ideal sheaf of smooth functions vanishing at  $t$ . The set  $\mathcal{R} := \pi^*\mathcal{C}_P^\infty$  is the sheaf of smooth functions on  $\widetilde{M}$  that are constant along the fibers, and it is naturally a subsheaf of  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ . Noting that  $\mathcal{R}/\pi^*\mathcal{I}_t$  is supported on  $M_t = \pi^{-1}(t)$ , we define

$$\mathcal{F}_t := \widetilde{\mathcal{F}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t.$$

Note that  $\widetilde{\mathcal{O}}_{\widetilde{M}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t = \mathcal{O}_{M_t}$ , the sheaf of holomorphic functions on  $M_t$ . Hence, if  $\mathcal{E}$  is an  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -module over  $\widetilde{M}$ , then  $\mathcal{E} \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t$  is an  $\mathcal{O}_{M_t}$ -module over  $M_t$ .

In particular, the sheaf  $\mathcal{F}_t$  is an  $\mathcal{O}_{M_t}$ -module. In adapted deformation coordinates, if  $X_1, \dots, X_r$  are local generators of  $\widetilde{\mathcal{F}}$ , given by

$$X_j(z^\alpha, t^\alpha) = \sum f_{ij}(z^\alpha, t^\alpha) \frac{\partial}{\partial z_i^\alpha},$$

then  $\mathcal{F}_{t_0}$  is locally generated by the  $X_j(z^\alpha, t_0^\alpha)$ 's. Namely it is generated by the vector fields

$$(3.2) \quad X_j(z^\alpha, t_0^\alpha) = \sum f_{ij}(z^\alpha, t_0^\alpha) \frac{\partial}{\partial z_i^\alpha}$$

obtained by evaluating  $f_{ij}(z^\alpha, t^\alpha)$  at  $t = t_0$ . From this remark, it follows easily:

**Lemma 3.2.** *For all  $t \in P$ , the sheaf  $\mathcal{F}_t$  defines a holomorphic foliation on  $M_t$ .*

We have the following exact sequence of  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -modules on  $\widetilde{M}$ :

$$(3.3) \quad 0 \longrightarrow \widetilde{\mathcal{F}} \longrightarrow \widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi) \longrightarrow \mathcal{N}_{\widetilde{\mathcal{F}}} \longrightarrow 0.$$

The *singular set* of  $\widetilde{\mathcal{F}}$  is by definition

$$S(\widetilde{\mathcal{F}}) := \{x \in \widetilde{M} : \mathcal{N}_{\widetilde{\mathcal{F}},x} \text{ is not } \mathcal{O}_{\widetilde{M},x} \text{ -- free}\}.$$

**Lemma 3.3.** *For each point  $x \in \widetilde{M}$  there exists an open neighborhood  $U \subset \widetilde{M}$  and two integers  $p, q \geq 0$  such that*

$$(3.4) \quad \widetilde{\mathcal{O}}_{\widetilde{M}}|_U^p \xrightarrow{\varphi} \widetilde{\mathcal{O}}_{\widetilde{M}}|_U^q \longrightarrow \mathcal{N}_{\widetilde{\mathcal{F}}}|_U \rightarrow 0,$$

*is an exact sequence of  $\widetilde{\mathcal{O}}_{\widetilde{M}}|_U$ -modules. Moreover,*

$$S(\widetilde{\mathcal{F}})|_U = \{x \in U : \text{rank } \varphi_x \text{ is not maximal}\}.$$

*Proof.* Since  $\widetilde{\mathcal{F}}$  is  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -coherent and  $\widetilde{\mathcal{O}}_{\widetilde{M}}(T\pi)$  is  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -locally free, from (3.3) it follows that  $\mathcal{N}_{\widetilde{\mathcal{F}}}$  is  $\widetilde{\mathcal{O}}_{\widetilde{M}}$ -coherent as well, so that (3.4) holds. The final statement follows from standard commutative algebra and (3.4).  $\square$

**Lemma 3.4.** *For each  $t \in P$  such that  $M_t \not\subset S(\tilde{\mathcal{F}})$  the following sequence of  $\mathcal{O}_{M_t}$ -modules over  $M_t$  is exact:*

$$(3.5) \quad 0 \rightarrow \tilde{\mathcal{F}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^* \mathcal{I}_t \rightarrow \tilde{\mathcal{O}}_{\tilde{M}}(T\pi) \otimes_{\mathcal{R}} \mathcal{R}/\pi^* \mathcal{I}_t \rightarrow \mathcal{N}_{\tilde{\mathcal{F}}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^* \mathcal{I}_t \rightarrow 0.$$

*Proof.* Since taking tensor products is right exact, it suffices to prove that the second map from the left is injective.

It is true on the stalk over each  $x \in M_t$  such that  $x \notin S(\tilde{\mathcal{F}})$ , since  $\mathcal{N}_{\tilde{\mathcal{F}},x}$  is  $\tilde{\mathcal{O}}_{\tilde{M},x}$ -free. We note that according to Lemma 3.3,  $S(\tilde{\mathcal{F}})|_{U \cap M_t} = \{x \in U \cap M_t : \text{rank } \varphi_x \text{ is not maximal}\}$ . Hence, for  $t$  fixed, these equations give rise to an analytic subset  $S(\tilde{\mathcal{F}}) \cap M_t$  of  $M_t$ , provided  $M_t \not\subset S(\tilde{\mathcal{F}})$ . As a consequence,  $S(\tilde{\mathcal{F}}) \cap M_t$  is thin in  $M_t$ . This shows that, since  $\tilde{\mathcal{F}}$  is a subsheaf of  $\tilde{\mathcal{O}}_{\tilde{M}}(T\pi)$ , it is also true on the stalk over  $x \in S(\tilde{\mathcal{F}}) \cap M_t$ .  $\square$

For each  $t \in P$  we have the following exact sequence of  $\mathcal{O}_{M_t}$ -modules:

$$(3.6) \quad 0 \longrightarrow \mathcal{F}_t \longrightarrow \mathcal{O}_{M_t}(TM_t) \longrightarrow \mathcal{N}_{\mathcal{F}_t} \longrightarrow 0.$$

**Definition 3.5.** Let  $t \in P$ . If  $M_t \subset S(\tilde{\mathcal{F}})$ , we let  $S(\mathcal{F}_t) := M_t$ . Otherwise we let

$$S(\mathcal{F}_t) := \{x \in M_t : \mathcal{N}_{\mathcal{F}_t,x} \text{ is not } \mathcal{O}_{M_t} \text{ -- free}\}.$$

**Proposition 3.6.** *For all  $t \in P$  it holds*

$$S(\mathcal{F}_t) = S(\tilde{\mathcal{F}}) \cap M_t.$$

*Proof.* If  $M_t \subset S(\tilde{\mathcal{F}})$  there is nothing to prove.

Thus, assume  $M_t \not\subset S(\tilde{\mathcal{F}})$ . Since  $\tilde{\mathcal{O}}_{\tilde{M}}(T\pi) \otimes_{\mathcal{R}} \mathcal{R}/\pi^* \mathcal{I}_t = \mathcal{O}_{M_t}(TM_t)$ , comparing (3.5) and (3.6) we see that

$$(3.7) \quad \mathcal{N}_{\mathcal{F}_t} = \mathcal{N}_{\tilde{\mathcal{F}}} \otimes_{\mathcal{R}} \mathcal{R}/\pi^* \mathcal{I}_t,$$

from which the statement follows at once.  $\square$

#### 4. RELATIVE BOTT VANISHING FOR A DEFORMATION OF FOLIATIONS

In this section we discuss a Bott type vanishing theorem for deformations of foliations. Thus, we let  $(\tilde{M}, P, \pi)$  be a deformation of manifolds and  $\tilde{\mathcal{F}}$  a deformation of foliations on  $\tilde{M}$ . In this section we assume

$$S(\tilde{\mathcal{F}}) = \emptyset$$

so that there exists an  $\tilde{\mathcal{O}}_{\tilde{M}}$ -subbundle  $\tilde{F}$  of  $T\pi$  such that  $\tilde{\mathcal{F}} = \tilde{\mathcal{O}}_{\tilde{M}}(\tilde{F})$ .

We refer to [3] for the notion of partial connections (see also [1], [2], [10]). As an example, given an  $\tilde{\mathcal{O}}_{\tilde{M}}$ -bundle  $E$  over  $\tilde{M}$ , we can define a “relative  $\bar{\partial}$ -connection” for  $E$  along  $\overline{T\pi}$  as follows. We let

$$\bar{\partial}_E : C_M^\infty(E) \rightarrow C_M^\infty(\overline{T^* \pi} \otimes E),$$

imposing that, given an  $\tilde{\mathcal{O}}_{\tilde{M}}$ -frame  $\{\sigma_1^\alpha, \dots, \sigma_r^\alpha\}$ , and a  $C^\infty$  section of  $E$ ,  $\sigma^\alpha := \sum f_j^\alpha \sigma_j^\alpha$ , it holds

$$\bar{\partial}_E(\sigma^\alpha) = \sum_{j=1}^r \sum_{k=1}^n \frac{\partial f_j^\alpha}{\partial \bar{z}_k^\alpha} d\bar{z}_k^\alpha \otimes \sigma_j^\alpha.$$

Since the transition matrices for  $E$  with respect to  $\tilde{\mathcal{O}}_{\tilde{M}}$ -frames contains only entries in  $\tilde{\mathcal{O}}_{\tilde{M}}$ , it is easy to see that such a definition is well given and it is a partial connection for  $E$  along  $\overline{T\pi}$ .

**Definition 4.1.** Let  $E$  be an  $\tilde{\mathcal{O}}_{\tilde{M}}$ -bundle over  $\tilde{M}$  and let  $\mathcal{E}$  be the sheaf of its  $\tilde{\mathcal{O}}_{\tilde{M}}$ -sections. A *flat partial  $\tilde{\mathcal{O}}_{\tilde{M}}$ -connection* for  $E$  along  $\tilde{\mathcal{F}}$  is a  $\mathbb{C}$ -linear map

$$\delta : \mathcal{E} \rightarrow \tilde{\mathcal{F}}^* \otimes \mathcal{E}$$

with the properties that for all  $X \in \tilde{\mathcal{F}}$ ,  $f, g \in \tilde{\mathcal{O}}_{\tilde{M}}$  and  $\sigma \in \mathcal{E}$

$$\delta_{(fX)}(g\sigma) = f(g\delta_X(\sigma) + dg(X)\sigma)$$

and

$$\delta_X \circ \delta_Y = 0, \quad \forall X, Y \in \tilde{\mathcal{F}}.$$

If  $\delta$  is as above, it induces a  $(C^\infty)$  partial connection

$$\delta : C_M^\infty(E) \rightarrow C_M^\infty(F^* \otimes E)$$

such that, for  $X \in \tilde{\mathcal{F}}$  and  $\sigma \in \mathcal{E}$ , we have  $\delta_X(\sigma) \in \mathcal{E}$ . Thus

$$\delta \oplus \bar{\partial}_E : C_M^\infty(E) \rightarrow C_M^\infty((F^* \oplus \overline{T^*\pi}) \otimes E)$$

is a partial connection. We say that a connection  $\nabla : C_M^\infty(E) \rightarrow C_M^\infty((T^*\tilde{M} \otimes \mathbb{C}) \otimes E)$  extends  $\delta \oplus \bar{\partial}_E$  if  $\nabla_X = (\delta \oplus \bar{\partial}_E)_X$  for all sections  $X$  of  $F \oplus \overline{T\pi}$ . Such a connection  $\nabla$  always exists (cf. [3]).

We have the following “relative Bott vanishing” theorem for actions of deformations of foliations:

**Theorem 4.2.** *let  $(\tilde{M}, P, \pi)$  be a deformation of manifolds and  $\tilde{\mathcal{F}}$  a deformation of foliations on  $\tilde{M}$  of rank  $p$ . Assume that  $S(\mathcal{F}) = \emptyset$ . Let  $\mathcal{E}$  be the sheaf of  $\tilde{\mathcal{O}}_{\tilde{M}}$ -sections of an  $\tilde{\mathcal{O}}_{\tilde{M}}$ -bundle  $E$  over  $\tilde{M}$ . Assume there exists a flat partial  $\tilde{\mathcal{O}}_{\tilde{M}}$ -connection  $\delta$  for  $\mathcal{E}$  along  $\tilde{\mathcal{F}}$ . Then, for any connection  $\nabla$  for  $E$  extending  $\delta \oplus \bar{\partial}_E$ , denoting by  $\iota_t : M_t \hookrightarrow \tilde{M}$  the natural embedding, it follows*

$$\iota_t^*(\varphi(\nabla)) = 0,$$

for all  $t \in P$  and all symmetric homogeneous polynomials  $\varphi$  of degree  $d > n - p$ .

*Proof.* Let  $\tilde{F}$  be the  $\tilde{\mathcal{O}}_{\tilde{M}}$ -bundle whose associated sheaf of sections is  $\tilde{\mathcal{F}}$ . Write

$$T\tilde{M} \otimes \mathbb{C} = \tilde{F} \oplus F_1 \oplus \overline{T\pi} \oplus \pi^*(TP),$$

where  $F_1$  is any  $C^\infty$  complement of  $\tilde{F}$  in  $T\pi$ .

Let  $K$  be the curvature of  $\nabla$ . Let  $\{s_1, \dots, s_p\}$  be a local  $\tilde{\mathcal{O}}_{\tilde{M}}$ -frame for  $\tilde{F}$ , and  $\{\frac{\partial}{\partial z_1}, \dots, \frac{\partial}{\partial z_n}\}$  the natural frame for  $\overline{T\pi}$  in adapted deformation coordinates. Since  $\tilde{F}$  is an  $\tilde{\mathcal{O}}_{\tilde{M}}$ -subbundle of  $T\pi$ , we can write  $s_j = \sum_{k=1}^n a_k(z, t) \frac{\partial}{\partial z_k}$  for  $j = 1, \dots, p$  and  $a_k \in \tilde{\mathcal{O}}_{\tilde{M}}$ . Hence,  $[s_j, \frac{\partial}{\partial z_k}] = 0$  for  $j = 1, \dots, p$  and  $k = 1, \dots, n$ .

Arguing similarly as in the proof of [3, Prop. 3.27] (see also [2, Thm. 6.1]) since  $\tilde{\mathcal{O}}_{\tilde{M}}$ -sections of  $E$  generate as  $C_M^\infty$ -module the sheaf of  $C^\infty$ -sections of  $E$ , one can see that

$$K(s_j, s_k) = K(s_j, \frac{\partial}{\partial \bar{z}_h}) = K(\frac{\partial}{\partial \bar{z}_h}, \frac{\partial}{\partial \bar{z}_l}) = 0$$

for all  $j, k = 1, \dots, p$  and  $h, l = 1, \dots, n$ . For instance, given  $\sigma$  an  $\tilde{\mathcal{O}}_{\tilde{M}}$ -section of  $E$ , we have

$$K(s_j, \frac{\partial}{\partial \bar{z}_h})(\sigma) = \nabla_{s_j}(\nabla_{\frac{\partial}{\partial \bar{z}_h}} \sigma) - \nabla_{\frac{\partial}{\partial \bar{z}_h}}(\nabla_{s_j} \sigma) - \nabla_{[s_j, \frac{\partial}{\partial \bar{z}_h}]} \sigma = 0,$$

because  $\nabla_{\frac{\partial}{\partial \bar{z}_h}} \sigma = (\bar{\partial}_E)_{\frac{\partial}{\partial \bar{z}_h}} \sigma = 0$  by definition, since  $\sigma$  is an  $\tilde{\mathcal{O}}_{\tilde{M}}$ -section;  $\nabla_{s_j} \sigma$  is another  $\tilde{\mathcal{O}}_{\tilde{M}}$ -section of  $E$ , hence  $\nabla_{\frac{\partial}{\partial \bar{z}_h}}(\nabla_{s_j} \sigma) = (\bar{\partial}_E)_{\frac{\partial}{\partial \bar{z}_h}}(\nabla_{s_j} \sigma) = 0$  and  $[s_j, \frac{\partial}{\partial \bar{z}_h}] = 0$ .

As a consequence, the entries of the matrix representing  $K$  are 2-forms belonging to the ideal generated by a dual basis of  $F_1$  (which has dimension  $n - p$ ) and by  $dt_1, \dots, dt_s$ , where these latter are a basis of  $\pi^*(T^*P)$ . Therefore, if  $\varphi$  has degree greater than  $n - p$ , it follows that

$$\varphi(\nabla) = \sum \omega_j \wedge dt_j,$$

for some  $2d - 1$  forms  $\omega_j$ , hence,  $\iota^*(\varphi(\nabla)) = 0$ .  $\square$

We recall that if  $M$  is a complex manifold and  $\mathcal{F}$  is a non-singular holomorphic foliation on  $M$  then there exists a natural holomorphic partial connection  $\delta$  for the normal bundle of the foliation  $\mathcal{N}_{\mathcal{F}}$  along  $\mathcal{F}$  given by the so called *Baum-Bott action* (see [3], [10]). Such a connection is *flat*, in the sense that  $\delta \circ \delta = 0$ . It is defined as follows:

$$(4.1) \quad \delta_X(\sigma) := \rho([X, \tilde{\sigma}])$$

where  $\sigma \in \mathcal{N}_{\mathcal{F}}$  is a holomorphic section of the normal bundle to the foliation,  $\tilde{\sigma} \in \mathcal{O}_M(TM)$  is a holomorphic section of the tangent bundle to  $M$  such that  $\rho(\tilde{\sigma}) = \sigma$ , where  $\rho : \mathcal{O}_M(TM) \rightarrow \mathcal{N}_{\mathcal{F}}$  is the natural projection, and  $X \in \mathcal{F}$ .

We are going to show that a deformation of foliations gives rise to a flat partial  $\tilde{\mathcal{O}}_{\tilde{M}}$ -connection for  $\mathcal{N}_{\tilde{\mathcal{F}}}$  along  $\tilde{\mathcal{F}}$  such that its “restriction” to each fiber  $M_t$  is the holomorphic flat partial connection for the normal bundle to  $\mathcal{F}_t$  given by the Baum-Bott action:

**Proposition 4.3.** *Let  $(\tilde{M}, P, \pi)$  be a deformation of manifolds and  $\tilde{\mathcal{F}}$  a deformation of foliations on  $\tilde{M}$ . Assume that  $S(\tilde{\mathcal{F}}) = \emptyset$ . Then there exists a flat partial  $\tilde{\mathcal{O}}_{\tilde{M}}$ -connection  $\tilde{\delta}$  for  $\mathcal{N}_{\tilde{\mathcal{F}}}$  along  $\tilde{\mathcal{F}}$ . Moreover, if  $\iota_t : M_t \hookrightarrow \tilde{M}$  is the natural embedding, then  $\iota_t^*(\tilde{\delta})$  is the holomorphic flat partial connection for  $\mathcal{N}_{\mathcal{F}}$  along  $\mathcal{F}_t$  given by the Baum-Bott action.*

*Proof.* Let  $\tilde{\rho} : \tilde{\mathcal{O}}_{\tilde{M}}(T\pi) \rightarrow \mathcal{N}_{\tilde{\mathcal{F}}}$  be the natural projection. For  $X \in \tilde{\mathcal{F}}$  and  $\sigma \in \mathcal{N}_{\tilde{\mathcal{F}}}$  we define

$$(4.2) \quad \tilde{\delta}_X(\sigma) := \tilde{\rho}([X, \tilde{\sigma}]),$$

where  $\tilde{\sigma} \in \tilde{\mathcal{O}}_{\tilde{M}}(T\pi)$  is such that  $\tilde{\rho}(\tilde{\sigma}) = \sigma$ . Involutivity of  $\tilde{\mathcal{F}}$  shows that  $\tilde{\delta}$  is well-defined and flatness follows from the Jacobi identity, so that  $\tilde{\delta}$  is a partial  $\tilde{\mathcal{O}}_{\tilde{M}}$ -connection for  $\mathcal{N}_{\tilde{\mathcal{F}}}$  along  $\tilde{\mathcal{F}}$ .

Comparing (4.2) with (4.1), it is easy to see that  $\iota_t^*(\tilde{\delta})$  is the flat partial  $\mathcal{O}_{M_t}$ -connection for  $\mathcal{N}_{\mathcal{F}_t}$  along  $\mathcal{F}_t$  given by the Baum-Bott action.  $\square$

In particular, Theorem 4.2 and Proposition 4.3 imply the following:

**Corollary 4.4.** *Let  $(\tilde{M}, P, \pi)$  be a deformation of manifolds and  $\tilde{\mathcal{F}}$  a deformation of foliations on  $\tilde{M}$ . Assume that  $S(\tilde{\mathcal{F}}) = \emptyset$ . Then there exists a connection  $\nabla$  for  $\mathcal{N}_{\tilde{\mathcal{F}}}$  such that, denoting by  $\iota_t : M_t \hookrightarrow \tilde{M}$  the natural embedding, it follows*

$$\iota_t^*(\varphi(\nabla)) = 0,$$

for all  $t \in P$  and all symmetric homogeneous polynomials  $\varphi$  of degree  $d > n - p$ .

## 5. RESIDUES OF BAUM-BOTT TYPES ON DEFORMATIONS OF MANIFOLDS

In this section we assume  $(\tilde{M}, P, \pi)$  is a deformation of manifolds and  $\tilde{\mathcal{F}}$  is a deformation of foliations on  $\tilde{M}$ . We also assume that  $\mathcal{N}_{\tilde{\mathcal{F}}}$  admits a  $C^\infty$  locally free resolution, namely, there exists an exact sequence of  $\mathcal{C}_M^\infty$ -modules:

$$(5.1) \quad 0 \rightarrow \mathcal{E}_q \rightarrow \cdots \rightarrow \mathcal{E}_0 \rightarrow \mathcal{N}_{\tilde{\mathcal{F}}} \otimes_{\tilde{\mathcal{O}}_{\tilde{M}}} \mathcal{C}_M^\infty \rightarrow 0,$$

such that each  $\mathcal{E}_j$  is locally  $\mathcal{C}_M^\infty$ -free.

*Remark 5.1.* If  $\tilde{\mathcal{F}}$  is locally  $\tilde{\mathcal{O}}_{\tilde{M}}$ -free then such a condition is satisfied with  $q = 1$  and  $\mathcal{E}_1 = \tilde{\mathcal{F}} \otimes_{\tilde{\mathcal{O}}_{\tilde{M}}} \mathcal{C}_M^\infty$ ,  $\mathcal{E}_0 = \tilde{\mathcal{O}}_{\tilde{M}}(T\pi) \otimes_{\tilde{\mathcal{O}}_{\tilde{M}}} \mathcal{C}_M^\infty$ .

Let  $E_j$  be the vector bundle over  $\tilde{M}$  whose sheaf of  $C^\infty$  sections is  $\mathcal{E}_j$ . Then  $\mathcal{N}_{\tilde{\mathcal{F}}}$  is a virtual bundle in the  $K$ -group  $K(\tilde{M})$  and its total Chern class is defined as

$$c(\mathcal{N}_{\tilde{\mathcal{F}}}) = \prod_{i=0}^q c(E_i)^{(-1)^i}.$$

We briefly sketch here the theory we need, and refer the reader to [3, Section 4], [7] and [10, Ch.II, 8] for details.

Let  $\tilde{U}_1$  be an open neighborhood of  $S(\tilde{\mathcal{F}})$  and let  $\tilde{U}_0 := \tilde{M} \setminus S(\tilde{\mathcal{F}})$ . We denote by  $(\nabla_0^\bullet, \nabla_1^\bullet)$  the family of  $q+1$  connections compatible with (5.1) and adapted to the covering  $\tilde{\mathcal{U}} := \{\tilde{U}_0, \tilde{U}_1\}$



of  $\widetilde{M}$ . Namely,  $\nabla_l^\bullet = (\nabla_l^q, \dots, \nabla_l^0)$ ,  $l = 0, 1$  is a family such that  $\nabla_l^j$  is a connection for  $E_j|_{\tilde{U}_l}$ ,  $j = 0, \dots, q$ ,  $l = 0, 1$  and the following diagram is commutative for  $i = 1, \dots, q$  and  $l = 0, 1$ :

$$(5.2) \quad \begin{array}{ccc} E_i|_{\tilde{U}_l} & \xrightarrow{\nabla_l^i} & C_M^\infty(T^*\widetilde{M} \otimes E_i|_{\tilde{U}_l}) \\ \downarrow & & \downarrow \\ E_{i-1}|_{\tilde{U}_l} & \xrightarrow{\nabla_l^{i-1}} & C_M^\infty(T^*\widetilde{M} \otimes E_{i-1}|_{\tilde{U}_l}) \end{array}$$

Moreover, let  $N_{\tilde{F}}$  be the vector bundle on  $\tilde{U}_0$  whose sheaf of sections is  $\mathcal{N}_{\tilde{F}} \otimes_{\tilde{\mathcal{O}}_{\tilde{M}}} C_M^\infty|_{\tilde{U}_0}$ . Let  $\nabla$  be an extension of the flat partial  $\tilde{\mathcal{O}}_{\tilde{M}}$ -connection  $\tilde{\delta}$  for  $\mathcal{N}_{\tilde{F}}|_{\tilde{U}_0}$  along  $\tilde{\mathcal{F}}$  given by Proposition 4.3. It is then possible to choose  $\nabla_0^\bullet$  to be compatible with  $\nabla$  (in the sense explained before).

Now, we let  $\varphi$  be a homogeneous symmetric polynomial of degree  $d > n - p$ . One can define the class  $\varphi(\mathcal{N}_{\tilde{F}})$  in the Čech-de Rham cohomology  $\check{H}^{2d}(\tilde{\mathcal{U}})$  which is represented by

$$\varphi(\nabla_*^\bullet) := (\varphi(\nabla_0^\bullet), \varphi(\nabla_1^\bullet), \varphi(\nabla_0^\bullet, \nabla_1^\bullet)),$$

where, by the compatibility condition,  $\varphi(\nabla_0^\bullet) = \varphi(\nabla)$  is a  $2d$  form on  $\tilde{U}_0$ ,  $\varphi(\nabla_1^\bullet)$  is the  $2d$  form on  $\tilde{U}_1$  associated to the family  $\nabla_1^\bullet$  and  $\varphi(\nabla_0^\bullet, \nabla_1^\bullet)$  is a  $(2d - 1)$ -form on  $\tilde{U}_0 \cap \tilde{U}_1$  such that  $d\varphi(\nabla_0^\bullet, \nabla_1^\bullet) = \varphi(\nabla_1^\bullet) - \varphi(\nabla_0^\bullet)$ . The Čech-de Rham cohomology  $\check{H}^*(\tilde{\mathcal{U}})$  is naturally isomorphic to the de Rham cohomology  $H_{dR}^*(\widetilde{M}, \mathbb{C})$ .

If  $M_t \not\subset S(\tilde{\mathcal{F}})$ , tensorizing (5.1) with  $\mathcal{R}/\pi^*\mathcal{I}_t$  we obtain the following exact sequence of  $C_{M_t}^\infty$ -modules (cf. the proof of Lemma 3.4):

$$(5.3) \quad 0 \rightarrow \mathcal{E}_q \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \rightarrow \dots \rightarrow \mathcal{E}_0 \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \rightarrow \mathcal{N}_{\tilde{\mathcal{F}}} \otimes_{\tilde{\mathcal{O}}_{\tilde{M}}} C_M^\infty \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t \rightarrow 0,$$

where  $\mathcal{E}_j \otimes_{\mathcal{R}} \mathcal{R}/\pi^*\mathcal{I}_t$  is the sheaf of  $C^\infty$  sections of the restriction of the bundle  $E_j$  to  $M_t$ . By (3.7), it is then easy to see the following:

**Lemma 5.2.** *Let  $t \in P$  and let  $\iota_t : M_t \rightarrow \widetilde{M}$  be the natural embedding. If  $M_t \not\subset S(\tilde{\mathcal{F}})$  then  $(\iota_t^*(\nabla_0^\bullet), \iota_t^*(\nabla_1^\bullet))$  is a family of connections for the virtual bundle  $\mathcal{N}_{\mathcal{F}_t}$  compatible with (5.3).*

By Corollary 4.4 and by the compatibility condition, it follows that for all homogeneous symmetric polynomials  $\varphi$  of degree  $d > n - p$ , the class  $\varphi(\mathcal{N}_{\mathcal{F}_t})$  is represented in the Čech-de Rham cohomology associated to the covering  $\tilde{\mathcal{U}} \cap M_t$  of  $M_t$  by the cocycle

$$\begin{aligned} \varphi(\iota_t^*\nabla_*^\bullet) &= (\iota_t^*\varphi(\nabla_0^\bullet), \iota_t^*\varphi(\nabla_1^\bullet), \iota_t^*\varphi(\nabla_0^\bullet, \nabla_1^\bullet)) = (\iota_t^*\varphi(\nabla), \iota_t^*\varphi(\nabla_1^\bullet), \iota_t^*\varphi(\nabla_0^\bullet, \nabla_1^\bullet)) \\ &= (0, \iota_t^*\varphi(\nabla_1^\bullet), \iota_t^*\varphi(\nabla_0^\bullet, \nabla_1^\bullet)). \end{aligned}$$

Hence, since by Proposition 3.6,  $\tilde{U}_0 \cap M_t = M_t \setminus S(\mathcal{F}_t)$ , the previous cocycle defines a localization of  $\varphi(\mathcal{N}_{\mathcal{F}_t})$ , call it  $\varphi(\mathcal{N}_{\mathcal{F}_t}, \mathcal{F}_t)$ , in the relative Čech-de Rham cohomology  $\check{H}^{2d}(\tilde{\mathcal{U}} \cap M_t, M_t \setminus S(\mathcal{F}_t))$ . The *Baum-Bott residue* is the image of  $\varphi(\mathcal{N}_{\mathcal{F}_t}, \mathcal{F}_t)$  by the Alexander homomorphism  $A : \check{H}^{2d}(\tilde{\mathcal{U}} \cap M_t, M_t \setminus S(\mathcal{F}_t)) \rightarrow H_{dR}^{2n-2d}(\tilde{U}_1 \cap M_t)^*$ . If  $S(\mathcal{F}_t)$  is made of  $k$  connected components then  $H_{dR}^{2n-2d}(\tilde{U}_1 \cap M_t)^*$  is a direct sum of  $k$  addends, and we can consider the Baum-Bott

residue at each connected component of  $S(\mathcal{F}_t)$ . If  $\tilde{U}_1 \cap M_t$  is a regular neighborhood of  $S(\mathcal{F}_t)$  then the above residue can be thought of as being in  $H_{2n-2d}(S(\mathcal{F}_t), \mathbb{C})$ .

Now, let  $S'(\tilde{\mathcal{F}}) \subseteq S(\tilde{\mathcal{F}})$  be a connected component. We assume that

$$S_t := M_t \cap S'(\tilde{\mathcal{F}}) \text{ is compact } \quad \forall t \in P.$$

*Remark 5.3.* Note that even if  $S(\tilde{\mathcal{F}})$  is connected by assumption,  $S(\mathcal{F}_t)$  might not.

Let  $\tilde{R}$  be a real manifold of dimension  $2n + s$  with boundary such that  $S'(\tilde{\mathcal{F}})$  is contained in the interior of  $\tilde{R}$ , no other components of  $S(\tilde{\mathcal{F}})$  intersect  $\tilde{R}$  and  $\partial\tilde{R}$  is transverse to  $M_t$  for all  $t \in P$ . Moreover, we can take  $\tilde{R}$  in such a way that  $R_t := \tilde{R} \cap M_t$  is compact for all  $t \in P$ .

We let  $U_t := \tilde{U}_1 \cap M_t$  and denote by  $H_{dR}^*(U_t)$  the de Rham cohomology of  $U_t$ . By the previous construction, we can express the *Baum-Bott residue*  $BB_\varphi(\mathcal{F}_t; S_t) \in H_{dR}^{2n-2d}(U_t)^*$  as follows:

$$(5.4) \quad BB_\varphi(\mathcal{F}_t; S_t) : H_{dR}^{2n-2d}(U_t) \ni [\tau] \mapsto \int_{R_t} \iota_t^* \varphi(\nabla_1^\bullet) \wedge \tau - \int_{\partial R_t} \iota_t^* \varphi(\nabla_0^\bullet, \nabla_1^\bullet) \wedge \tau.$$

*Remark 5.4.* Note that there exists a natural morphism  $H_{dR}^{2n-2d}(U_t)^* \rightarrow H_{dR}^{2n-2d}(M_t)^*$ . Therefore one can remove the dependence on  $\tilde{U}_1$  in this construction. Moreover, if  $M_t$  is compact, then  $H_{dR}^{2n-2d}(M_t)^* = H_{2n-2d}(M_t)$ .

Now we are in good shape to prove our main result:

**Theorem 5.5.** *Let  $(\tilde{M}, P, \pi)$  be a deformation of manifolds and  $\tilde{\mathcal{F}}$  a deformation of foliations on  $\tilde{M}$  of rank  $p$ . Suppose that  $\mathcal{N}_{\tilde{\mathcal{F}}}$  admits a  $C^\infty$  locally free resolution. Let  $S'(\tilde{\mathcal{F}}) \subseteq S(\tilde{\mathcal{F}})$  be a connected component of the singular set of  $\tilde{\mathcal{F}}$  and let  $S_t := M_t \cap S'(\tilde{\mathcal{F}})$ . Assume that for all  $t \in P$  the set  $S_t$  is compact and  $S_t \neq M_t$ . Let  $\varphi$  be a homogeneous symmetric polynomial of degree  $d > n - p$ . Under these assumptions, the Baum-Bott residue  $BB_\varphi(\mathcal{F}_t; S_t)$  is continuous in  $t \in P$ . Namely, for any  $C^\infty$   $(2n - 2d)$ -form  $\tilde{\tau}$  on  $\tilde{M}$  such that  $\iota_t^*(\tilde{\tau})$  is closed for all  $t \in P$ ,*

$$\lim_{t \rightarrow t_0} BB_\varphi(\mathcal{F}_t; S_t) (\iota_t^*(\tilde{\tau})) = BB_\varphi(\mathcal{F}_{t_0}; S_{t_0}) (\iota_{t_0}^*(\tilde{\tau})).$$

*Proof.* From the previous construction and (5.4) it follows that the Baum-Bott residues on  $M_t$  are expressed by means of smooth forms on  $\tilde{M}$ . Hence, they vary continuously.  $\square$

Note that, if  $S_t$  is not connected and  $S_t = \cup_\lambda S_t^\lambda$  is its connected components decomposition, then

$$BB_\varphi(\mathcal{F}_t; S_t) = \sum_\lambda BB_\varphi(\mathcal{F}_t; S_t^\lambda).$$

## 6. AN EXAMPLE

In  $\mathbb{P}^3$  with homogeneous coordinates  $[x_1 : x_2 : x_3 : x_4]$  we consider the vector field which is defined in the affine chart  $x_4 \neq 0$  with coordinates  $x = x_1/x_4, y = x_2/x_4, z = x_3/x_4$  by

$$X(x, y, z) := x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + z \frac{\partial}{\partial z}.$$

The singularities are the line  $L$  given by  $x_1 = x_2 = 0$  and the point at infinity given by  $Q := [1 : 1 : 1 : 0]$  (see the next expression (6.2)).

The vector field  $X$  generates a one-dimensional foliation  $\mathcal{F}$  given by  $X : \mathbb{P}^3 \times \mathbb{C} \rightarrow T\mathbb{P}^3$  on  $\mathbb{P}^3$ . By the Baum-Bott theorem, we can localize  $\varphi(T\mathbb{P}^3/\mathcal{F})$  for homogeneous symmetric polynomials  $\varphi$  of degree 3. Such polynomials are essentially given by  $c_1^3, c_1c_2$  and  $c_3$ . Moreover, since  $\mathcal{F}$  is trivial, we obtain that  $\varphi(T\mathbb{P}^3/\mathcal{F}) = \varphi(T\mathbb{P}^3)$ . Let  $\mathcal{O}(1)$  be the hyperplane bundle on  $\mathbb{P}^3$  and let  $\xi := c_1(\mathcal{O}(1)) \in H_{dR}^2(\mathbb{P}^3)$ . From the Euler exact sequence, it follows that  $c(T\mathbb{P}^3) = (1 + \xi)^4$ , from which

$$(6.1) \quad \int_{\mathbb{P}^3} c_1^3(T\mathbb{P}^3) = 64, \quad \int_{\mathbb{P}^3} c_1c_2(T\mathbb{P}^3) = 24, \quad \int_{\mathbb{P}^3} c_3(T\mathbb{P}^3) = 4.$$

Changing coordinates, in the affine chart  $x_3 \neq 0$  with coordinates  $\tilde{x} = x_1/x_3, \tilde{y} = x_2/x_3, \tilde{z} = x_4/x_3$  the vector field  $X$  has the expression:

$$(6.2) \quad X(\tilde{x}, \tilde{y}, \tilde{z}) = (\tilde{x} - \tilde{x}\tilde{y}) \frac{\partial}{\partial \tilde{x}} + (\tilde{x} - \tilde{y}^2) \frac{\partial}{\partial \tilde{y}} - \tilde{y}\tilde{z} \frac{\partial}{\partial \tilde{z}}.$$

From this it follows that the first jet of  $X$  at  $Q$  is given by the non-degenerate matrix

$$A := \begin{pmatrix} 0 & -1 & 0 \\ 1 & -2 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Hence since  $Q$  is a non-degenerate isolated singularity for  $X$  it follows (see, e.g. [3, (0.7)] or [10])

$$(6.3) \quad \text{BB}_\varphi(X; Q) = \frac{\varphi(A)}{\det A},$$

that is

$$(6.4) \quad \text{BB}_{c_1^3}(X; Q) = 27 \quad \text{BB}_{c_1c_2}(X; Q) = 9 \quad \text{BB}_{c_3}(X; Q) = 1.$$

By the Baum-Bott theorem,

$$\int_{\mathbb{P}^3} \varphi(T\mathbb{P}^3) = \text{BB}_\varphi(X; Q) + \text{BB}_\varphi(X; L).$$

From this and by (6.1) and (6.4) we obtain

$$(6.5) \quad \text{BB}_{c_1^3}(X; L) = 37 \quad \text{BB}_{c_1c_2}(X; L) = 15 \quad \text{BB}_{c_3}(X; L) = 3.$$

However, computing such residues directly without using the Baum-Bott theorem seem to be very complicated because the singular set is not isolated.

We present now a deformation procedure which allows to compute the previous residues and explain in practice how our Theorem 1.1 works.

Let  $\widetilde{M} := \mathbb{P}^3 \times (-1, 1)$  and let  $\widetilde{\mathcal{F}}$  be the deformation of foliations defined by the vector fields  $X_t$ ,  $t \in (-1, 1)$ , which on the chart  $x_4 \neq 0$  are defined as

$$X_t(x, y, z) = (x + tz) \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + y \frac{\partial}{\partial z}.$$

On the chart  $x_3 \neq 0$  the vector field  $X_t$  is given by

$$X(\tilde{x}, \tilde{y}, \tilde{z}) = (\tilde{x} - \tilde{x}\tilde{y} + t) \frac{\partial}{\partial \tilde{x}} + (\tilde{x} - \tilde{y}^2) \frac{\partial}{\partial \tilde{y}} - \tilde{y}\tilde{z} \frac{\partial}{\partial \tilde{z}}.$$

The singularities of  $X_t$  for  $t \neq 0$  are given by  $O := [0 : 0 : 0 : 1]$  and  $P_j(t) := [u_{t,j}^2 : u_{t,j} : 1 : 0]$  for  $j = 1, 2, 3$ , where the  $u_{t,j}$ 's are the three roots of the equation  $\lambda^3 - \lambda^2 - t = 0$ .

At the point  $O$  the first jet of  $X_t$ ,  $t \neq 0$ , is non-degenerate and it is given by the matrix

$$\begin{pmatrix} 1 & 0 & t \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

From this it and from (6.3),

$$(6.6) \quad \text{BB}_{c_1^3}(X_t; O) = \frac{1}{t} \quad \text{BB}_{c_1 c_2}(X_t; O) = 0 \quad \text{BB}_{c_3}(X_t; O) = 1.$$

*Remark 6.1.* It is interesting to note that  $\lim_{t \rightarrow 0} \text{BB}_{c_1^3}(X_t; O) = \infty$ , namely, the residue by itself is not continuous, but it is so when taken the sum of the residues for the singularities which belong to the same connected components in the ambient space  $\widetilde{M}$ .

At the point  $P_j(t)$  the vector field  $X_t$  has first jet given by the matrix

$$B(t, j) := \begin{pmatrix} 1 - u_{t,j} & -u_{t,j}^2 & 0 \\ 1 & -2u_{t,j} & 0 \\ 0 & 0 & -u_{t,j} \end{pmatrix},$$

with determinant  $\det B(t, j) = u_{t,j}^2(2 - 3u_{t,j})$ . Thus, for  $t \rightarrow 0$ ,  $t \neq 0$  the points  $P_j(t)$  are isolated non-degenerate singularities for  $X_t$  and one can use (6.3) to compute the residues:

$$(6.7) \quad \begin{aligned} \text{BB}_{c_1^3}(X_t; P_j(t)) &= \frac{(1 - 4u_{t,j})^3}{u_{t,j}^2(2 - 3u_{t,j})} & \text{BB}_{c_1 c_2}(X_t; P_j(t)) &= \frac{3(6u_{t,j} - 1 - 8u_{t,j})}{u_{t,j}^2(2 - 3u_{t,j})} \\ \text{BB}_{c_3}(X_t; P_j(t)) &= 1. \end{aligned}$$

Now, as  $t \rightarrow 0$ , it follows that two of the roots of the equation  $\lambda^3 - \lambda^2 - t = 0$  tend to 0 and one tends to 1. We assume that  $u_{t,1}, u_{t,2} \rightarrow 0$  and  $u_{t,3} \rightarrow 1$ . Hence, if  $S'(\widetilde{\mathcal{F}})$  is the connected component which contains the line  $L$  in the manifold deformation  $M \times (-1, 1)$ , the intersection

of  $S'(\tilde{\mathcal{F}})$  with  $M \times \{t\}$  is given by the points  $O, P_1(t), P_2(t)$ . While, the connected component in  $M \times (-1, 1)$  which contains  $Q$  contains all the points  $P_3(t)$ .

A direct computation—taking into account that  $u_{t,1} + u_{t,2} + u_{t,3} = 1$ ,  $u_{t,1}u_{t,2} + u_{t,1}u_{t,3} + u_{t,2}u_{t,3} = 0$  and  $u_{t,1}u_{t,2}u_{t,3} = t$ —shows that for  $\varphi = c_1^3, c_1c_2, c_3$

$$\lim_{t \rightarrow 0} \text{BB}_\varphi(X_t; P_3(t)) = \text{BB}_\varphi(X; Q),$$

$$\lim_{t \rightarrow 0} [\text{BB}_\varphi(X_t; P_1(t)) + \text{BB}_\varphi(X_t; P_2(t)) + \text{BB}_\varphi(X_t; O)] = \text{BB}_\varphi(X; L).$$

## REFERENCES

- [1] M. Abate, F. Bracci, T. Suwa and F. Tovena, *Localization of Atiyah classes*, arXiv:1005.1482v1 [math.CV], 2010.
- [2] M. Abate, F. Bracci and F. Tovena, *Index theorems for holomorphic maps and foliations*, Indiana Univ. Math. J. **57** (2008), 2999–3048.
- [3] P. Baum and R. Bott, *Singularities of holomorphic foliations*, J. Differential Geom. **7** (1972), 279–342.
- [4] El H. M. Dia, *Quelques résultats nouveaux sur les résidus de Baum-Bott*, Thèse, Université Montpellier II, 2007.
- [5] X. Gómez-Mont, *The transverse dynamics of a holomorphic flow*, Ann. of Math. (2) **127** (1988), no. 1, 49–92.
- [6] K. Kodaira and D. C. Spencer, *On deformations of complex analytic structures. I, II*, Ann. of Math. (2) **67** (1958), 328–466.
- [7] D. Lehmann, *Variétés stratifiées  $C^\infty$ : intégration de Čech-de Rham, et théorie de Chern-Weil*, Geometry and topology of submanifolds, II (Avignon, 1988), 205–248, World Sci. Publ., Teaneck, NJ, 1990.
- [8] G. Pourcin, *Deformations of singular holomorphic foliations on reduced compact  $\mathbb{C}$ -analytic spaces*, Holomorphic dynamics (Mexico, 1986), 246–255, Lecture Notes in Math., 1345, Springer, Berlin, 1988.
- [9] H.-J. Reiffen, *The variety of a moduli of foliations on a complex space*, Enseign. Math. (2) **33** (1987), no. 3–4, 191–197.
- [10] T. Suwa, *Indices of Vector Fields and Residues of Singular Holomorphic Foliations*, Actualités Mathématiques, Hermann, Paris, 1998.

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